KEY CONCEPT *Factoring Perfect Square Trinomials*

Words If a trinomial can be written in the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$, then it can be factored as $(a + b)^2$ or as $(a - b)^2$, respectively.

Symbols $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$

Example: Factor Perfect Square Trinomials

$9y^2 - 12y + 4$

1. Is the first term a perfect square? Yes, $9y^2 = (3y)^2$.
2. Is the last term a perfect square? Yes, $4 = 2^2$.
3. Is the middle term equal to $2(3y)(2)$? Yes, $12y = 2(3y)(2)$.

$9y^2 - 12y + 4$ is a perfect square trinomial.

$9y^2 - 12y + 4 = (3y)^2 - 2(3y)(2) + 2^2$ Write as $a^2 - 2ab + b^2$.

$= (3y - 2)^2$ Factor using the pattern.

Check your progress:

1) $9x^2 - 12x + 4$

Step 1: Product $= 9 \cdot 4 = 36$

Sum $= -12$

We need to find numbers that multiply to 36 and add to -12. Because the product is positive and the sum is negative, both factors must be negative.

Step 2:

Factors of 36:	Sum of factors:
-1, -36	$-1 + -36 = -37$
-2, -18	$-2 + -18 = -20$
-3, -12	$-3 + -12 = -15$
-4, -9	$-4 + -9 = -13$
-6, -6	$-6 + -6 = -12$

-6 and -6 are the factors that will make a product of 36 and a sum of -12.

Step 3:

$9x^2 - 12x + 4$

$9x^2 - 6x - 6x + 4$

Step 4:

$9x^2 - 6x \,\big\{\, - 6x + 4$

The first two terms are divisible by $3x$. The last two terms are divisible by -2. Note that I'm choosing a negative two for the last two terms because the first term in that binomial is negative.

$3x\left(\frac{9x^2}{3x} - \frac{6x}{3x}\right) \big\{ - 2\left(\frac{-6x}{-2} + \frac{4}{-2}\right)$

$3x(3x - 2) - 2(3x - 2)$

Both terms have a $(3x - 2)$, so we can factor that out.

$(3x - 2)\left(\frac{3x(3x - 2)}{3x - 2} - \frac{2(3x - 2)}{3x - 2}\right)$

$(3x - 2)(3x - 2)$ **Because both of these factors are exactly the same, we can write this as a square of the factor.

$(3x - 2)^2$ **This is our solution

2) $16x^2 + 24x + 9$

Step 1: Product $= 16 \cdot 9 = 144$

Sum $= 24$

We need to find numbers that multiply to 144 and add to 24. Because the product is positive and the sum is negative, both factors must be negative.

Step 2:

Factors of 144:	Sum of factors:
1, 144	$1 + 144 = 145$
2, 72	$2 + 72 = 74$
3, 48	$3 + 48 = 51$
4, 36	$4 + 36 = 40$
6, 24	$6 + 24 = 30$
8, 18	$8 + 18 = 26$
9, 16	$9 + 16 = 25$
12, 12	$12 + 12 = 24$

12 and 12 are the factors that will make a product of 144 and a sum of 24.

Step 3:

$16x^2 + 24x + 9$

$16x^2 + 12x + 12x + 9$

Step 4:

$16x^2 + 12x + 12x + 9$

The first two terms are divisible by $4x$. The last two terms are divisible by 3. Note that I'm choosing a positive three for the last two terms because the first term in that binomial is positive.

$$4x\left(\frac{16x^2}{4x} + \frac{12x}{4x}\right) + 3\left(\frac{12x}{3} + \frac{9}{3}\right)$$

$4x(4x + 3) + 3(4x + 3)$

Both terms have a $(4x + 3)$, so we can factor that out.

$$(4x + 3)\left(\frac{4x(4x + 3)}{4x + 3} + \frac{3(4x + 3)}{4x + 3}\right)$$

$(4x + 3)(4x + 3)$ **Because both of these factors are exactly the same, we can write this as a square of the factor.

$(4x + 3)^2$ **This is our solution

<u>Example: Factor Completely</u>

Factor each polynomial.

a. $4x^2 - 36$

First check for a GCF. Then, since the polynomial has two terms, check for the difference of squares.

$4x^2 - 36 = 4(x^2 - 9)$ — 4 is the GCF.

$= 4(x^2 - 3^2)$ — $x^2 = x \cdot x$ and $9 = 3 \cdot 3$

$= 4(x + 3)(x - 3)$ — Factor the difference of squares.

b. $25x^2 + 5x - 6$

This is not a perfect square trinomial. It is of the form $ax^2 + bx + c$. Are there two numbers m and n with a product of $25(-6)$ or -150 and a sum of 5? Yes, the product of 15 and -10 is -150 and the sum is 5.

$25x^2 + 5x - 6 = 25x^2 + mx + nx - 6$ — Write the pattern.

$= 25x^2 + 15x - 10x - 6$ — $m = 15$ and $n = -10$

$= (25x^2 + 15x) + (-10x - 6)$ — Group terms with common factors.

$= 5x(5x + 3) - 2(5x + 3)$ — Factor out the GCF from each grouping.

$= (5x + 3)(5x - 2)$ — $5x + 3$ is the common factor.

Check your progress:

1) $2x^2 + 18$

Both factors are divisible by 2, so let's factor that out.

$$2\left(\frac{2x^2}{2} + \frac{18}{2}\right)$$

$2(x^2 + 9)$ **Since what is left in parentheses is a sum of squares, this is as far as this problem factors and this is our solution.

2) $c^2 - 5c + 6$

Step 1: Product $= 1 \cdot 6 = 6$

Sum $= -5$

**We need to find numbers that multiply to 6 and add to -5. Because the product is positive and the sum is negative, that means both factors must be negative.

Step 2:

Factors of 6:	Sum of factors:
-1, -6	$-1 + -6 = -7$
-2, -3	$-2 + -3 = -5$

-2 and -3 are the factors that will make a product of 6 and a sum of -5.

Step 3:

$$c^2 - 5c + 6$$

$$c^2 - 2c - 3c + 6$$

Step 4:

$$c^2 - 2c \;\Big|\; -3c + 6$$

The first two terms are divisible by c. The last two terms are divisible by -3. Note that I'm choosing a negative three for the last two terms because the first term in that binomial is negative.

$$c\left(\frac{c^2}{c} - \frac{2c}{c}\right) \Big| -3\left(\frac{-3c}{-3} + \frac{6}{-3}\right)$$

$c(c-2) - 3(c-2)$

Both terms have a $(c-2)$, so we can factor that out.

$(c-2)\left(\frac{c(c-2)}{c-2}-\frac{3(c-2)}{c-2}\right)$

$(c-2)(c-3)$ **This is our solution

3) $5a^3 - 80a$

If we look at this binomial, we can see that both terms are divisible by $5a$. So, let's factor that out, first.

$5a\left(\frac{5a^3}{5a}-\frac{80a}{5a}\right)$

$5a(a^2-36)$

We should recognize that the binomial in the parentheses is a difference of squares. So, we should rewrite the binomial in parentheses so that each term is a square.

$5a((a)^2-(6)^2)$

Now, we factor the difference of squares.

$5a(a+6)(a-6)$ **This is our solution.

4) $8x^2 - 18x - 35$

Step 1: Product $= 8 \cdot -35 = -280$

Sum $= -18$

We need to find numbers that multiply to -280 and add to -18. Because the product is negative, one of the factors must be negative. Because the sum is negative, the larger factor must be negative.

Step 2:

Factors of -280:	Sum of factors:
1, -280	$1 + -280 = -279$
2, -140	$2 + -140 = -138$
4, -70	$4 + -70 = -66$
5, -56	$5 + -56 = -51$
7, -40	$7 + -40 = -33$
8, -35	$8 + -35 = -27$

10, -28	$10 + -28 = -18$
14, -20	$14 + -20 = -6$

10 and -28 are the factors that will make a product of -280 and a sum of -18.

Step 3:

$$8x^2 - 18x - 35$$

$$8x^2 + 10x - 28x - 35$$

Step 4:

$$8x^2 + 10x - 28x - 35$$

The first two terms are divisible by $2x$. The last two terms are divisible by -7. Note that I'm choosing a negative seven for the last two terms because the first term in that binomial is negative.

$$2x\left(\frac{8x^2}{2x} + \frac{10x}{2x}\right) - 7\left(\frac{-28x}{-7} + \frac{-35}{-7}\right)$$

$$2x(4x + 5) - 7(4x + 5)$$

Both terms have a $(4x + 5)$, so we can factor that out.

$$(4x + 5)\left(\frac{2x(4x + 5)}{4x + 5} - \frac{7(4x + 5)}{4x + 5}\right)$$

$(4x + 5)(2x - 7)$ **This is our solution

5) $9g^2 + 12g - 4$

Step 1: Product $= 9 \cdot -4 = -36$

Sum $= 12$

We need to find numbers that multiply to -36 and add to 12. Because the product is negative, ***one* of the factors must be negative. Because sum is positive, that means the larger factor must be positive.

Step 2:

Factors of -36:	Sum of factors:
-1, 36	$-1 + 36 = 35$
-2, 18	$-2 + 18 = 16$

-3, 12 $-3 + 12 = 9$

-4, 9 $-4 + 9 = 5$

-6, 6 $-6 + 6 = 0$

We have found all factors of -36 and none of them add to 12, so the polynomial is prime and cannot be factored.

Prime **This is our solution

6) $3m^3 + 2m^2n - 12m - 8n$

Remember that to factor a polynomial with four terms, we split the polynomial in half and treat it like two binomials we are factoring separately.

$3m^3 + 2m^2n - 12m - 8n$

The first two terms are divisible by m^2. The last two terms are divisible by -4.

$m^2\left(\frac{3m^3}{m^2} + \frac{2m^2n}{m^2}\right) - 4\left(\frac{-12m}{-4} + \frac{-8n}{-4}\right)$

$m^2(3m + 2n) - 4(3m + 2)$

Both terms have a $(3m + 2n)$, so we can factor that out.

$(3m + 2n)\left(\frac{m^2(3m + 2n)}{3m + 2n} - \frac{4(3m + 2n)}{3m + 2n}\right)$

$(3m + 2n)(m^2 - 4)$

At this point, we should notice that one of our factors is a difference of squares $(m^2 - 4)$.

Rewrite the term that is factorable so that each term of the binomials is a square. We leave everything else alone.

$(3m + 2n)((m)^2 - (2)^2)$

Then we factor the last binomial.

$(3m + 2n)(m + 2)(m - 2)$ **This is our solution.

Example: Solve Equations with Repeated Factors

Solve $x^2 - x + \frac{1}{4} = 0$.

$x^2 - x + \frac{1}{4} = 0$	Original equation
$x^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 = 0$	Recognize $x^2 - x + \frac{1}{4}$ as a perfect square trinomial.
$\left(x - \frac{1}{2}\right)^2 = 0$	Factor the perfect square trinomial.
$x - \frac{1}{2} = 0$	Set repeated factor equal to zero.
$x = \frac{1}{2}$	Solve for x.

Check your progress:

1) $4x^2 + 4x + 1 = 0$

Because one side of the equation is already equal to zero, we can move directly to factoring the trinomial.

Factor: $4x^2 + 4x + 1 = 0$

Step 1: Product $= 4 \cdot 1 = 4$

Sum $= 4$

**We need to find numbers that multiply to 4 and add to 4. Because the product is positive and the sum is positive, both of the factors must be positive

Step 2:

Factors of 4:	Sum of factors:
1, 4	$1 + 4 = 5$
2, 2	$2 + 2 = 4$

2 and 2 are the factors that will make a product of 4 and a sum of 4.

Step 3:

$4x^2 + 4x + 1 = 0$

$4x^2 + 2x + 2x + 1 = 0$

Step 4:

$4x^2 + 2x + 2x + 1 = 0$

The first two terms are divisible by $2x$. The last two terms are divisible by 1.

$$2x\left(\frac{4x^2}{2x}+\frac{2x}{2x}\right)+1\left(\frac{2x}{1}+\frac{1}{1}\right)=0$$

$$2x(2x+1)+1(2x+1)=0$$

Both terms have an $(2x+1)$, so we can factor that out.

$$(2x+1)\left(\frac{2x(2x+1)}{2x+1}+\frac{1(2x+1)}{2x+1}\right)=0$$

$$(2x+1)(2x+1)=0$$

$(2x+1)^2=0$ **Because both terms are the same, we should write this as a square.

We will use the square root property to solve.

$$(2x+1)^2=0$$

$$\sqrt{(2x+1)^2}=\sqrt{0}$$

$$2x+1=0$$

$$\quad -1 \quad -1$$

$$2x=-1$$

$$\frac{2x}{2}=\frac{-1}{2}$$

$$x=-\frac{1}{2}$$

A perfect square trinomial will only have one solution, in this case it is $x=-\frac{1}{2}$.

Now, let's check our solution and make sure it works.

Check:

Replace x with -1/2

$$4x^2+4x+1=0$$

$$4\left(-\tfrac{1}{2}\right)^2+4\left(-\tfrac{1}{2}\right)+1=0$$

$4\left(\frac{1}{4}\right) - \frac{4}{2} + 1 = 0$

$\frac{4}{4} - 2 + 1 = 0$

$1 - 1 = 0$

$0 = 0$

Because the equation is true, we know that we have found the correct solution.

2) $4x^2 - 28x = -49$

We need to get one side of the equation equal to zero to solve.

$4x^2 - 28x = -49$

$+49 \quad + 49$

$4x^2 - 28x + 49 = 0$

Factor: $4x^2 - 28x + 49 = 0$

Step 1: Product $= 4 \cdot 49 = 196$

Sum $= -28$

**We need to find numbers that multiply to 196 and add to -28. Because the product is positive and the sum is negative, both of the factors must be negative.

Step 2:

Factors of 196:	Sum of factors:
-1, -196	$-1 + -196 = -197$
-2, -98	$-2 + -98 = -100$
-4, -49	$-4 + -49 = -53$
-14, -14	$-14 + -14 = -28$

-14 and -14 are the factors that will make a product of 196 and a sum of -28.

Step 3:

$4x^2 - 28x + 49 = 0$

$4x^2 - 14x - 14x + 49 = 0$

Step 4:

$4x^2 - 14x - 14x + 49 = 0$

The first two terms are divisible by $2x$. The last two terms are divisible by -7.

$2x\left(\frac{4x^2}{2x} - \frac{14x}{2x}\right) - 7\left(\frac{-14x}{-7} + \frac{49}{-7}\right) = 0$

$2x(2x - 7) - 7(2x - 7) = 0$

Both terms have a $(2x - 7)$, so we can factor that out.

$(2x - 7)\left(\frac{2x(2x - 7)}{2x - 7} - \frac{7(2x - 7)}{2x - 7}\right) = 0$

$(2x - 7)(2x - 7) = 0$

$(2x - 7)^2 = 0$ **Because both terms are the same, we should write this as a square.

We will use the square root property to solve.

$(2x - 7)^2 = 0$

$\sqrt{(2x - 7)^2} = \sqrt{0}$

$2x - 7 = 0$

$\quad +7 \quad +7$

$2x = 7$

$\frac{2x}{2} = \frac{7}{2}$

$x = \frac{7}{2}$

A perfect square trinomial will only have one solution, in this case it is $x = \frac{7}{2}$.

Now, let's check our solution and make sure it works.

Check:

Replace x with 7/2

$4x^2 - 28x = -49$

$4\left(\frac{7}{2}\right)^2 - 28\left(\frac{7}{2}\right) = -49$

$$4\left(\frac{49}{4}\right) - \frac{196}{2} = -49$$

$$\frac{196}{4} - 98 = -49$$

$$49 - 98 = -49$$

$$-49 = -49$$

Because the equation is true, we know that we have found the correct solution.