## KEY CONCEPT

## Factoring Perfect Square Trinomials

Words If a trinomial can be written in the form $a^{2}+2 a b+b^{2}$ or $a^{2}-2 a b+b^{2}$, then it can be factored as $(a+b)^{2}$ or as $(a-b)^{2}$, respectively.
Symbols $a^{2}+2 a b+b^{2}=(a+b)^{2}$ and $a^{2}-2 a b+b^{2}=(a-b)^{2}$

## Example: Factor Perfect Square Trinomials

$$
9 y^{2}-12 y+4
$$

(1) Is the first term a perfect square? Yes, $9 y^{2}=(3 y)^{2}$.
(2) Is the last term a perfect square? Yes, $4=2^{2}$.
(3) Is the middle term equal to $2(3 y)(2)$ ? Yes, $12 y=2(3 y)(2)$.
$9 y^{2}-12 y+4$ is a perfect square trinomial.
$9 y^{2}-12 y+4=(3 y)^{2}-2(3 y)(2)+2^{2}$ Write as $a^{2}-2 a b+b^{2}$.

$$
=(3 y-2)^{2} \quad \text { Factor using the pattern. }
$$

Check your progress:

1) $9 x^{2}-12 x+4$

Step 1: $\quad$ Product $=9 \cdot 4=36$

$$
\text { Sum }=-12
$$

We need to find numbers that multiply to 36 and add to -12 . Because the product is positive and the sum is negative, both factors must be negative.

Step 2:
Factors of 36: Sum of factors:
-1, -36

$$
-1+-36=-37
$$

$-2,-18$
$-2+-18=-20$
$-3,-12$
$-3+-12=-15$
$-4,-9$
$-4+-9=-13$
$-6,-6$
$-6+-6=-12$
-6 and -6 are the factors that will make a product of 36 and a sum of -12 .
Step 3:
$9 x^{2}-12 x+4$
$9 x^{2}-6 x-6 x+4$

Step 4:
$\left.9 x^{2}-6 x\right\}-6 x+4$
The first two terms are divisible by $3 x$. The last two terms are divisible by -2 . Note that I'm choosing a negative two for the last two terms because the first term in that binomial is negative.
$\left.3 x\left(\frac{9 x^{2}}{3 x}-\frac{6 x}{3 x}\right)\right\}-2\left(\frac{-6 x}{-2}+\frac{4}{-2}\right)$
$3 x(3 x-2)-2(3 x-2)$
Both terms have a $(3 x-2)$, so we can factor that out.
$(3 x-2)\left(\frac{3 x(3 x-2)}{3 x-2}-\frac{2(3 x-2)}{3 x-2}\right)$
$(3 x-2)(3 x-2) \quad * *$ Because both of these factors are exactly the same, we can write this as a square of the factor.
$(3 x-2)^{2} \quad * *$ This is our solution
2) $16 x^{2}+24 x+9$

Step 1: $\quad$ Product $=16 \cdot 9=144$
Sum $=24$
We need to find numbers that multiply to 144 and add to 24 . Because the product is positive and the sum is negative, both factors must be negative.

Step 2:
Factors of 144: Sum of factors:
$1,144 \quad 1+144=145$
$2,72 \quad 2+72=74$
3, 48
$3+48=51$
4,36
$4+36=40$
6, 24
$6+24=30$
8, 18
$8+18=26$
9, 16
$9+16=25$
12, 12
$12+12=24$
12 and 12 are the factors that will make a product of 144 and a sum of 24 .

Step 3:

$$
\begin{aligned}
& 16 x^{2}+24 x+9 \\
& 16 x^{2}+12 x+12 x+9
\end{aligned}
$$

Step 4:
$16 x^{2}+12 x\{+12 x+9$
The first two terms are divisible by $4 x$. The last two terms are divisible by 3 . Note that I'm choosing a positive three for the last two terms because the first term in that binomial is positive.
$\left.4 x\left(\frac{16 x^{2}}{4 x}+\frac{12 x}{4 x}\right)\right\}+3\left(\frac{12 x}{3}+\frac{9}{3}\right)$
$4 x(4 x+3)+3(4 x+3)$
Both terms have a $(4 x+3)$, so we can factor that out.
$(4 x+3)\left(\frac{4 x(4 x+3)}{4 x+3}+\frac{3(4 x+3)}{4 x+3}\right)$
$(4 x+3)(4 x+3) \quad * *$ Because both of these factors are exactly the same, we can write this as a square of the factor.
$(4 x+3)^{2} \quad * *$ This is our solution

## Example: Factor Completely

## Factor each polynomial.

a. $4 x^{2}-36$

First check for a GCF. Then, since the polynomial has two terms, check for the difference of squares.

$$
\begin{aligned}
4 x^{2}-36 & =4\left(x^{2}-9\right) & & 4 \text { is the GCF. } \\
& =4\left(x^{2}-3^{2}\right) & & x^{2}=x \cdot x \text { and } 9=3 \cdot 3 \\
& =4(x+3)(x-3) & & \text { Factor the difference of squares. }
\end{aligned}
$$

b. $25 x^{2}+5 x-6$

This is not a perfect square trinomial. It is of the form $a x^{2}+b x+c$. Are there two numbers $m$ and $n$ with a product of $25(-6)$ or -150 and a sum of 5 ? Yes, the product of 15 and -10 is -150 and the sum is 5 .

$$
\begin{aligned}
25 x^{2}+5 x-6 & =25 x^{2}+m x+n x-6 & & \text { Write the pattern. } \\
& =25 x^{2}+15 x-10 x-6 & & m=15 \text { and } n=-10 \\
& =\left(25 x^{2}+15 x\right)+(-10 x-6) & & \text { Group terms with common factors. } \\
& =5 x(5 x+3)-2(5 x+3) & & \text { Factor out the GCF from each grouping. } \\
& =(5 x+3)(5 x-2) & & 5 x+3 \text { is the common factor. }
\end{aligned}
$$

## Check your progress:

1) $2 x^{2}+18$

Both factors are divisible by 2 , so let's factor that out.
$2\left(\frac{2 x^{2}}{2}+\frac{18}{2}\right)$
$2\left(x^{2}+9\right) \quad * *$ Since what is left in parentheses is a sum of squares, this is as far as this problem factors and this is our solution.
2) $c^{2}-5 c+6$

Step 1: $\quad$ Product $=1 \cdot 6=6$

$$
\text { Sum }=-5
$$

**We need to find numbers that multiply to 6 and add to -5 . Because the product is positive and the sum is negative, that means both factors must be negative.

Step 2:
Factors of 6: Sum of factors:
$\begin{array}{ll}-1,-6 & -1+-6=-7 \\ -2,-3 & -2+-3=-5\end{array}$
-2 and -3 are the factors that will make a product of 6 and a sum of -5 .
Step 3:

$$
\begin{gathered}
c^{2}-5 c+6 \\
c^{2} \stackrel{-2 c-3 c}{ }+6
\end{gathered}
$$

Step 4:
$c^{2}-2 c\{3 c+6$
The first two terms are divisible by $c$. The last two terms are divisible by -3 . Note that I'm choosing a negative three for the last two terms because the first term in that binomial is negative.
$c\left(\frac{c^{2}}{c}-\frac{2 c}{c}\right)\left\{-3\left(\frac{-3 c}{-3}+\frac{6}{-3}\right)\right.$
$c(c-2)-3(c-2)$
Both terms have a $(c-2)$, so we can factor that out.
$(c-2)\left(\frac{c(c-2)}{c-2}-\frac{3(c-2)}{c-2}\right)$
$(c-2)(c-3) \quad * *$ This is our solution
3) $5 a^{3}-80 a$

If we look at this binomial, we can see that both terms are divisible by $5 a$. So, let's factor that out, first.
$5 a\left(\frac{5 a^{3}}{5 a}-\frac{80 a}{5 a}\right)$
$5 a\left(a^{2}-16\right)$
We should recognize that the binomial in the parentheses is a difference of squares. So, we should rewrite the binomial in parentheses so that each term is a square.
$5 a\left((a)^{2}-(4)^{2}\right)$
Now, we factor the difference of squares.
$5 a(a+4)(a-4) \quad * *$ This is our solution.
4) $8 x^{2}-18 x-35$

Step 1: $\quad$ Product $=8 \cdot-35=-280$

$$
\text { Sum }=-18
$$

We need to find numbers that multiply to -280 and add to -18 . Because the product is negative, one of the factors must be negative. Because the sum is negative, the larger factor must be negative.

Step 2:
Factors of -280: Sum of factors:
$1,-280 \quad 1+-280=-279$
$2,-140 \quad 2+-140=-138$
4, -70
$4+-70=-66$
5, -56
$5+-56=-51$
7, -40
$7+-40=-33$
8, -35
$8+-35=-27$

10, -28
$10+-28=-18$
14, -20
$14+-20=-6$
10 and -28 are the factors that will make a product of -280 and a sum of -18 .
Step 3:
$8 x^{2}-18 x-35$
$8 x^{2}+10 x-28 x-35$

Step 4:
$\left.8 x^{2}+10 x\right\} 28 x-35$
The first two terms are divisible by $2 x$. The last two terms are divisible by -7 . Note that I'm choosing a negative seven for the last two terms because the first term in that binomial is negative.
$\left.2 x\left(\frac{8 x^{2}}{2 x}+\frac{10 x}{2 x}\right)\right\}-7\left(\frac{-28 x}{-7}+\frac{-35}{-7}\right)$
$2 x(4 x+5)-7(4 x+5)$
Both terms have a $(4 x+5)$, so we can factor that out.
$(4 x+5)\left(\frac{2 x(4 x+5)}{4 x+5}-\frac{7(4 x+5)}{4 x+5}\right)$
$(4 x+5)(2 x-7) \quad * *$ This is our solution
5) $9 g^{2}+12 g-4$

Step 1: $\quad$ Product $=9 \cdot-4=-36$

$$
\text { Sum }=12
$$

**We need to find numbers that multiply to -36 and add to 12 . Because the product is negative, one of the factors must be negative. Because sum is positive, that means the larger factor must be positive.

Step 2:
Factors of -36: Sum of factors:
-1, 36
$-1+36=35$
$-2,18$
$-2+18=16$
$-3,12 \quad-3+12=9$
$-4,9 \quad-4+9=5$
$-6,6 \quad-6+6=0$
We have found all factors of -36 and none of them add to 12 , so the polynomial is prime and cannot be factored.

Prime $\quad * *$ This is our solution
6) $3 m^{3}+2 m^{2} n-12 m-8 n$

Remember that to factor a polynomial with four terms, we split the polynomial in half and treat it like two binomials we are factoring separately.
$3 m^{3}+2 m^{2} n\{-12 m-8 n$
The first two terms are divisible by $m^{2}$. The last two terms are divisible by -4 .
$m^{2}\left(\frac{3 m^{3}}{m^{2}}+\frac{2 m^{2} n}{m^{2}}\right)-4\left(\frac{-12 m}{-4}+\frac{-8 n}{-4}\right)$
$m^{2}(3 m+2 n)-4(3 m+2)$
Both terms have a $(3 m+2 n)$, so we can factor that out.
$(3 m+2 n)\left(\frac{m^{2}(3 m+2 n)}{3 m+2 n}-\frac{4(3 m+2 n)}{3 m+2 n}\right)$
$(3 m+2 n)\left(m^{2}-4\right)$
At this point, we should notice that one of our factors is a difference of squares $\left(m^{2}-4\right)$.
Rewrite the term that is factorable so that each term of the binomials is a square. We leave everything else alone.
$(3 m+2 n)\left((m)^{2}-(2)^{2}\right)$
Then we factor the last binomial.
$(3 m+2 n)(m+2)(m-2) \quad * * T h i s ~ i s ~ o u r ~ s o l u t i o n . ~$

## Example: Solve Equations with Repeated Factors

Solve $x^{2}-x+\frac{1}{4}=0$.

$$
x^{2}-x+\frac{1}{4}=0 \quad \text { Original equation }
$$

$$
x^{2}-2(x)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^{2}=0 \quad \text { Recognize } x^{2}-x+\frac{1}{4} \text { as a perfect square trinomial. }
$$

$$
\left(x-\frac{1}{2}\right)^{2}=0 \quad \text { Factor the perfect square trinomial. }
$$

$$
x-\frac{1}{2}=0 \quad \text { Set repeated factor equal to zero. }
$$

$$
x=\frac{1}{2} \quad \text { Solve for } x .
$$

## Check your progress:

1) $4 x^{2}+4 x+1=0$

Because one side of the equation is already equal to zero, we can move directly to factoring the trinomial.

Factor: $4 x^{2}+4 x+1=0$
Step 1: $\quad$ Product $=4 \cdot 1=4$

$$
\text { Sum }=4
$$

**We need to find numbers that multiply to 4 and add to 4 . Because the product is positive and the sum is positive, both of the factors must be positive

Step 2:
Factors of 4: Sum of factors:
$1,4 \quad 1+4=5$
2, 2
$2+2=4$
2 and 2 are the factors that will make a product of 4 and a sum of 4 .

Step 3:

$$
\begin{gathered}
4 x^{2}+4 x+1=0 \\
4 x^{2}+2 x+2 x+1=0
\end{gathered}
$$

Step 4:
$\left.4 x^{2}+2 x\right\}+2 x+1=0$

The first two terms are divisible by $2 x$. The last two terms are divisible by 1 .
$\left.2 x\left(\frac{4 x^{2}}{2 x}+\frac{2 x}{2 x}\right)\right\}+1\left(\frac{2 x}{1}+\frac{1}{1}\right)=0$
$2 x(2 x+1)+1(2 x+1)=0$
Both terms have an $(2 x+1)$, so we can factor that out.
$(2 x+1)\left(\frac{2 x(2 x+1)}{2 x+1}+\frac{1(2 x+1)}{2 x+1}\right)=0$
$(2 x+1)(2 x+1)=0$
$(2 x+1)^{2}=0 \quad * *$ Because both terms are the same, we should write this as a square

We will use the square root property to solve.
$(2 x+1)^{2}=0$
$\sqrt{(2 x+1)^{2}}=\sqrt{0}$
$2 x+1=0$
$-1 \quad-1$
$2 x=-1$
$\frac{2 x}{2}=\frac{-1}{2}$
$x=-\frac{1}{2}$
A perfect square trinomial will only have one solution, in this case it is $x=-\frac{1}{2}$.

Now, let's check our solution and make sure it works.

Check:
Replace x with $-1 / 2$

$$
\begin{aligned}
& 4 x^{2}+4 x+1=0 \\
& 4\left(-\frac{1}{2}\right)^{2}+4\left(-\frac{1}{2}\right)+1=0
\end{aligned}
$$

$$
\begin{aligned}
& 4\left(\frac{1}{4}\right)-\frac{4}{2}+1=0 \\
& \frac{4}{4}-2+1=0 \\
& 1-1=0 \\
& 0=0
\end{aligned}
$$

Because the equation is true, we know that we have found the correct solution.
2) $4 x^{2}-28 x=-49$

We need to get one side of the equation equal to zero to solve.
$4 x^{2}-28 x=-49$

$$
+49+49
$$

$4 x^{2}-28 x+49=0$
Factor: $4 x^{2}-28 x+49=0$
Step 1: $\quad$ Product $=4 \cdot 49=196$

$$
\text { Sum }=-28
$$

**We need to find numbers that multiply to 196 and add to -28 . Because the product is positive and the sum is negative, both of the factors must be negative.

Step 2:
Factors of 196: Sum of factors:
-1, -196
-2, -98
-4, -49
$-14,-14$
$-1+-196=-197$
$-2+-98=-100$
$-4+-49=-53$
$-14+-14=-28$
-14 and -14 are the factors that will make a product of 196 and a sum of -28 .

Step 3:

$$
\begin{gathered}
4 x^{2}-28 x+49=0 \\
4 x^{2} \overbrace{-14 x}^{-14 x}+49=0
\end{gathered}
$$

Step 4:
$4 x^{2}-14 x\{-14 x+49=0$
The first two terms are divisible by $2 x$. The last two terms are divisible by -7 .
$\left.2 x\left(\frac{4 x^{2}}{2 x}-\frac{14 x}{2 x}\right)\right\}-7\left(\frac{-14 x}{-7}+\frac{49}{-7}\right)=0$
$2 x(2 x-7)-7(2 x-7)=0$
Both terms have a $(2 x-7)$, so we can factor that out.
$(2 x-7)\left(\frac{2 x(2 x-7)}{2 x-7}-\frac{7(2 x-7)}{2 x-7}\right)=0$
$(2 x-7)(2 x-7)=0$
$(2 x-7)^{2}=0 \quad * *$ Because both terms are the same, we should write this as a square.

We will use the square root property to solve.
$(2 x-7)^{2}=0$
$\sqrt{(2 x-7)^{2}}=\sqrt{0}$
$2 x-7=0$

$$
+7 \quad+7
$$

$2 x=7$
$\frac{2 x}{2}=\frac{7}{2}$
$x=\frac{7}{2}$
A perfect square trinomial will only have one solution, in this case it is $x=\frac{7}{2}$.
Now, let's check our solution and make sure it works.
Check:
Replace x with $7 / 2$

$$
\begin{aligned}
& 4 x^{2}-28 x=-49 \\
& 4\left(\frac{7}{2}\right)^{2}-28\left(\frac{7}{2}\right)=-49
\end{aligned}
$$

$$
\begin{aligned}
& 4\left(\frac{49}{4}\right)-\frac{196}{2}=-49 \\
& \frac{196}{4}-98=-49 \\
& 49-98=-49 \\
& -49=-49
\end{aligned}
$$

Because the equation is true, we know that we have found the correct solution.

